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HEAT TRANSFER BY LAMINAR FORCED FLOW AGAINST A NON-ISOTHERMAL ROTATING DISK

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Abstract—Theoretical consideration is given to the temperature distributions and heat-transfer results in laminar forced flow against a non-isothermal rotating disk. The surface temperature of the disk is assumed to vary according to a power law with the radial distance. Numerical solutions of the boundary-layer equations are presented to indicate the effects of the forced flow, of the non-uniform surface temperature and of the Prandtl number. Asymptotic heat-transfer relations for large and small Prandtl numbers are given.

NOMENCLATURE

- a, flow constant defined in (5), s^{-1};
- A, dimensionless parameter defined in (12);
- b, flow constant defined in (5), ft/s;
- B, dimensionless parameter defined in (12);
- C, dimensionless parameter defined in (14);
- F, G, H, dimensionless functions defined in (6);
- h_r, local heat-transfer coefficient at r, Btu/ft²s degF;
- k, thermal conductivity of fluid, Btu/ft s degF;
- m, dimensionless constant defined in (5);
- n, constant defined in (5), degF/ft^m;
- Nu_r, local Nusselt number at r;
- Pr, Prandtl number of fluid;
- r, radial co-ordinate, ft;
- T, temperature, °F;
- T_w, wall temperature, °F;
- T_∞, free-stream temperature, °F;
- u, velocity component in r-direction, ft/s;
- v, velocity component in ϕ-direction, ft/s;
- w, velocity component in z-direction, ft/s;
- z, co-ordinate normal to disk surface, ft.

Greek symbols

- α, thermal diffusivity of fluid, ft²/s;
- δ, dimensionless thermal boundary-layer thickness;
- ζ, dimensionless distance, η/δ;
- η, dimensionless distance defined in (7);
- θ, dimensionless temperature defined in (6);
- λ, dimensionless parameter defined in (7);
- ν, kinematic viscosity of fluid, ft²/s;
- ϕ, azimuthal co-ordinate, rad;
- ω, angular velocity in ϕ-direction, rad/s.

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INTRODUCTION

THE flow and heat transfer about a rotating disk has long been a subject of investigations. The laminar flow and heat transfer about a rotating disk situated in a large body of quiescent fluid were first analysed by von Kármán [1] and by Millsaps and Pohlhausen [2], respectively. In recent years, considerable attention has been given to the extensions of this problem such as the effects of compressibility [3], of Prandtl number [4], and of non-uniform surface temperature [5, 6]. The more general problem of a forced flow against a rotating disk, however, has received relatively little attention. The general nature of this problem can be fully understood by realizing its two limiting cases, the flow induced by a rotating disk (i.e. without forced flow), and the axisymmetrical stagnation flow (i.e. without rotation of the disk).

The exact solution for the laminar forced flow against a rotating disk was first given by Hannah [7]. Independently, the exact solution was rediscovered by Tifford and Chu [8], and an approximate solution obtained from the integral method was given by Schlichting and Truckenbrodt [9]. More recently, Yamaga [10] extended the integral analysis [9] to the heat-transfer problem. But his results are quite limited and their accuracy is unknown.

In the present paper, consideration is given to the extension of the exact analysis for the laminar forced flow against a rotating disk to the heat-transfer problem in such a flow. The analysis is based on a power-law wall-temperature distribution, with uniform wall temperature as a special case. Numerical results obtained are presented to indicate the effects of the forced flow, of the non-uniform wall temperature and of the Prandtl number. Asymptotic heat-transfer relations for large and small Prandtl numbers are also given.

ANALYSIS

The physical system under consideration is a steady, incompressible, laminar flow over an infinitely extended disk rotating at a constant angular speed about its symmetrical axis. There exists no external force field in the system. The fluid is assumed to have constant physical properties. The co-ordinate system is shown in Fig. 1. The governing equations for such a system, under boundary-layer approximations and negligible viscous dissipation, are given as:

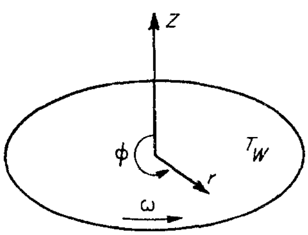

FIG. 1. The co-ordinate system.

continuity:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

momentum:

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = \nu\frac{\partial^2 u}{\partial z^2} + \left(u\frac{\partial u}{\partial r}\right)_{z\to\infty} \tag{2}$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\frac{\partial^2 v}{\partial z^2} \tag{3}$$

energy:

$$u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = \alpha\frac{\partial^2 T}{\partial z^2}. \tag{4}$$

The boundary conditions are:

$$\left.\begin{array}{ll} u(r,0) = 0 & u(r,\infty) = ar \\ v(r,0) = \omega r & v(r,\infty) = 0 \\ w(r,0) = 0 & w(r,\infty) = -2az - b \\ T(r,0) = T_\infty + nr^m & T(r,\infty) = T_\infty. \end{array}\right\} \tag{5}$$

The surface temperature of the disk is assumed to follow a power-law distribution due to similarity consideration. When $m = 0$, the surface temperature is uniform. The free-stream velocity boundary conditions are obtained from the potential-flow solution [11].

The partial differential system can be reduced to an ordinary one if the following transformations are introduced:

$$\left.\begin{array}{l} u = \lambda r F(\eta),\quad v = \omega r G(\eta),\quad w = (\nu\lambda)^{1/2} H(\eta) \\ T - T_\infty = (T_w - T_\infty)\,\theta(\eta) \end{array}\right\} \tag{6}$$

where

$$\lambda = (a^2 + \omega^2)^{1/2}, \quad \eta = (\lambda/\nu)^{1/2} z. \tag{7}$$

The governing equations after the above transformation become

$$H' + 2F = 0 \tag{8}$$

$$F'' - H F' = F^2 - A^2 G^2 - B^2 \tag{9}$$

$$G'' - H G' - 2F G = 0 \tag{10}$$

$$\theta'' - (Pr) H\theta' - (m\,Pr) F\theta = 0 \tag{11}$$

where

$$A = \omega/\lambda, \quad B = a/\lambda. \tag{12}$$

The boundary conditions are given as

$$\left.\begin{array}{ll} F(0)=0 & F(\infty)=B \\ G(0)=1 & G(\infty)=0 \\ H(0)=0 & H(\infty)=-2B\eta-C \\ \theta(0)=1 & \theta(\infty)=0 \end{array}\right\} \quad (13)$$

where

$$C=b(\nu\lambda)^{-1/2}. \quad (14)$$

The first three equations, (8), (9) and (10) with their corresponding boundary conditions were integrated numerically by Hannah [7] for cases with $(A/B)=0$, $\frac{1}{2}$, 1, 2 and ∞. The case with $(A/B)=0$ represents the axisymmetrical stagnation flow, which has been studied by Homann [11]. The other limiting case with $(A/B)=\infty$ is the induced flow due to a rotating disk, and this was first integrated numerically by Cochran [12]. For the heat-transfer part, (11) with its boundary conditions has been studied extensively for the case with $(A/B)=\infty$ [2–6]. The heat-transfer results for $(A/B)=0$ and $m=0$ were given by Sibulkin [13]. For values of (A/B) other than the above two limits, Yamaga [10] obtained the heat-transfer results for $m=0$ (isothermal disk) and $Pr=0{\cdot}72$ and 1·0 by use of integral analysis.

In the present analysis, the velocity functions H and F reported by Hannah [7] were used to obtain the temperature function θ in (11) by use of IBM 7090 electronic computer. The resulting dimensionless profiles are shown in Figs. 2–5 for $m=0$, $Pr=10$, 1·0 and 0·1, and for $m=2$, and $Pr=1{\cdot}0$.

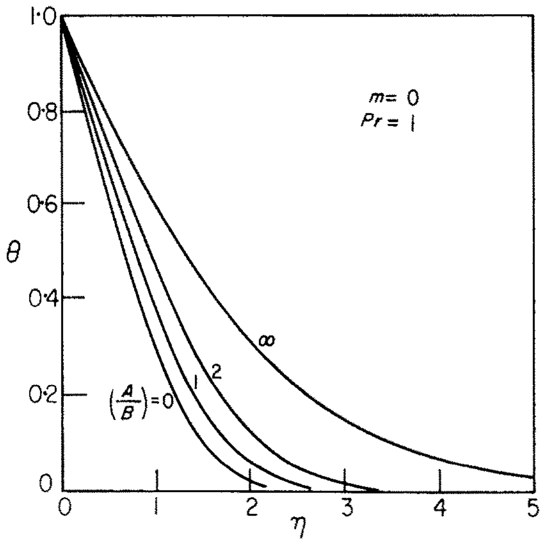

FIG. 3. Distributions of dimensionless temperature ($m=0$, $Pr=1$).

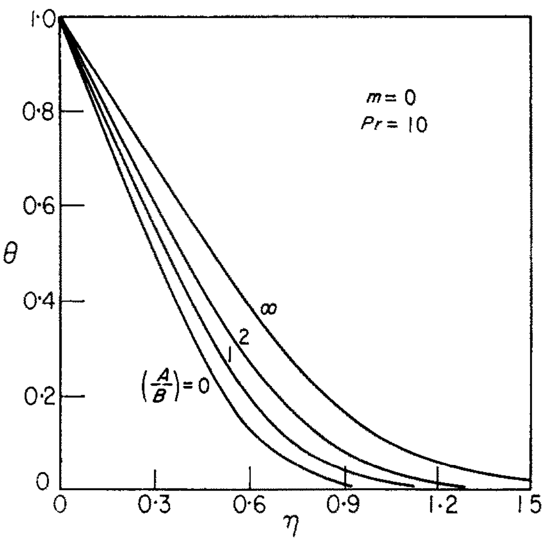

FIG. 2. Distributions of dimensionless temperature ($m=0$, $Pr=10$).

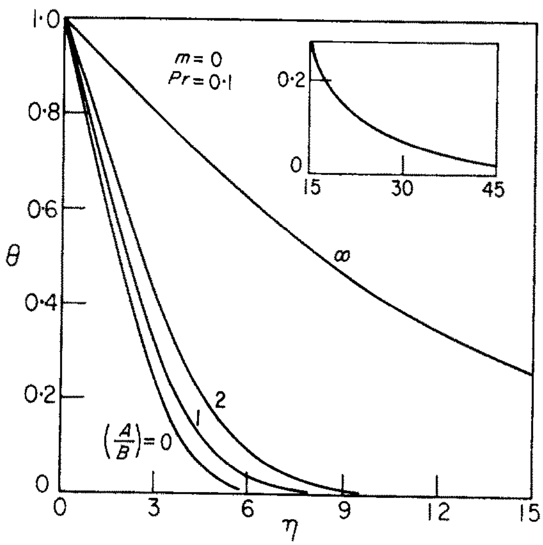

FIG. 4. Distributions of dimensionless temperature ($m=0$, $Pr=0{\cdot}1$).

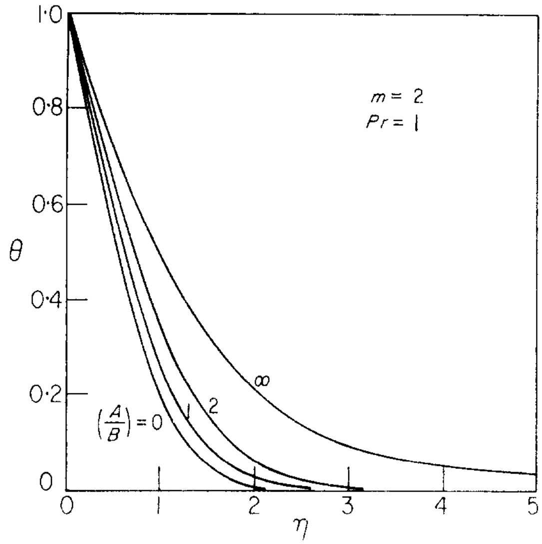

FIG. 5. Distributions of dimensionless temperature ($m = 2$, $Pr = 1$).

HEAT-TRANSFER RESULTS

From the definition of local heat-transfer coefficient, h_r:

$$-k\left(\frac{\partial T}{\partial z}\right)_{z=0} \equiv h_r\,(T_w - T_\infty) \qquad (15)$$

there follows

$$h_r = -k\left(\frac{\lambda}{\nu}\right)^{1/2}\theta'(0). \qquad (16)$$

Therefore, a local Nusselt number may be conveniently defined as

$$Nu_r \equiv \frac{h_r}{k}\left(\frac{\nu}{\lambda}\right)^{1/2} = -\theta'(0). \qquad (17)$$

The local Nusselt numbers for different cases are tabulated in Table 1. In the special case of an isothermal disk ($m = 0$), the local Nusselt numbers from exact analysis are compared with those available for $Pr = 1{\cdot}0$ and $Pr = 10$ from integral analysis [10]. It is shown that the integral analysis by Yamaga gives the heat-transfer results with an error less than 10 per cent from the exact results of boundary-layer equations.

ASYMPTOTIC HEAT-TRANSFER RELATIONS

First, as the Prandtl number becomes very small, the velocity boundary layer is very thin as compared with the thermal boundary layer. Therefore, in solving the energy equation (11), the following approximation can be made:

$$H(\eta) \simeq H(\infty) = -2B\eta - C \qquad (18)$$

and

$$F(\eta) \simeq F(\infty) = B \qquad (19)$$

where values of B and C depend upon the ratio of (A/B) as tabulated in Table 2. The energy equation thus becomes

$$\theta'' + (Pr)(2B\eta + C)\,\theta' - (m\,B\,Pr)\,\theta = 0 \quad (20)$$

with the boundary conditions prescribed in (13). No general solution of closed form is available.

The special case of an isothermal disk ($m = 0$), however, does possess a solution. It can be easily shown that for $B \neq 0$ the local Nusselt number is

$$Nu_r = (Pr\,B)^{1/2}\exp\left(-Pr\,C^2/4B\right)\{1 - \mathrm{erf}\,[(B\,Pr)^{1/2}\,C/2B]\}^{-1}. \qquad (21)$$

For $B = 0$, i.e. the induced flow due to a rotating disk, the solution has been given in [4].

Table 1. Values of the local Nusselt number

$\frac{A}{B}$	Nu_r ($m = 0$, $Pr = 10$)	Nu_r ($m = 0$, $Pr = 1{\cdot}0$)	Nu_r ($m = 0$, $Pr = 0{\cdot}1$)	Nu_r ($m = 2$, $Pr = 1{\cdot}0$)
0	1·752 (1·800)*	0·762 (0·780)	0·301	1·075
1	1·535 (1·562)	0·658 (0·677)	0·257	0·934
2	1·340 (1·361)	0·557 (0·564)	0·210	0·800
∞	1·134 (1·170)	0·396 (0·364)	0·077	0·616

* Values in parentheses are from integral analysis [10].

Table 2. Flow constants for forced flow against a rotating disk from [7]

$\frac{A}{B}$	B	C	$H''(0)$
0	1·000	−0·569	−2·624
1	0·707	−0·440	−1·872
2	0·447	−0·275	−1·372
∞	0·000	0·886	−1·020

An approximate solution to (20) can be obtained by applying the integral concept only to the thermal boundary layer. This integral analysis for asymptotic heat-transfer results is slightly different from that of Yamaga, in which both the velocity and thermal boundary layers are subject to general integral consideration. Consider a fourth-order polynomial profile,

$$\theta = 1 - 2\zeta + 2\zeta^3 - \zeta^4 \qquad (22)$$

where $\zeta \equiv \eta/\delta$, and δ is the thermal boundary-layer thickness. This temperature profile satisfies the appropriate boundary conditions. Substituting (22) into (20) and integrating (20) across the thermal boundary layer, there follows for $B = 0$:

$$Nu_r = C\,Pr \qquad (23)$$

and for $B \neq 0$:

$$Nu_r = \frac{1{\cdot}2\,B\,Pr\,(m+2)}{[(Pr\,C)^2 + 2{\cdot}4\,B\,Pr\,(m+2)]^{1/2} - C\,Pr}. \qquad (24)$$

It is interesting to note that the dependence of m is missing in (23). This indicates that for the induced flow due to a rotating disk, the local Nusselt number at very low Prandtl numbers is independent of any specific power-law wall-temperature distribution. This is also a direct consequence of (20) as the effect of m disappears at $B = 0$.

When the Prandtl number is large, the thermal boundary layer is affected only by a very thin region of the velocity boundary layer near the wall. Thus the velocity field in the thermal boundary layer is given by

$$H(\eta) = H''(0)\,\eta^2/2 \qquad (25)$$

and from (8),

$$F(\eta) = -H''(0)\,\eta/2 \qquad (26)$$

where the value of $H''(0)$ is a function of (A/B) as given in Table 2. Consequently, the energy equation becomes

$$\theta'' - \tfrac{1}{2}\,Pr\,H''(0)\,\eta^2\theta' + \tfrac{1}{2}\,m\,Pr\,H''(0)\,\eta\theta = 0 \qquad (27)$$

with the temperature boundary conditions in (13).

The solution to (27) for an isothermal disk ($m = 0$) gives

$$Nu_r = [-Pr\,H''(0)/6]^{1/3}/\Gamma(4/3) \qquad (28)$$

which is of the same form as given in [4] for the induced flow due to a rotating disk ($A/B = \infty$). For a non-isothermal disk of power-law wall-temperature distribution, the integral analysis

Table 3. Asymptotic heat-transfer results from (23), (24) *and* (29)

$\frac{A}{B}$	Nu_r ($m = 0, Pr = 100$)	Nu_r ($m = 0, Pr = 0{\cdot}01$)	Nu_r ($m = 2, Pr = 100$)	Nu_r ($m = 2, Pr = 0{\cdot}01$)
0	4·12 (3·94)*	0·109 (0·097)	5·19	0·156
1	3·68 (3·52)	0·0904 (0·0810)	4·64	0·128
2	3·32 (3·18)	0·0718 (0·0654)	4·18	0·103
∞	3·01 [2·69]†	0·00886 [0·00871]	3·79 [3·71]	0·00886

* Values in parentheses are from solutions given in (21) and (28).
† Values in brackets are from the numerical solution in [4, 6].

based on the temperature profile in (22) gives

$$Nu_r = 2\left[-\frac{(m+2)\,Pr\,H''(0)}{60}\right]^{1/3}. \qquad (29)$$

The approximate asymptotic heat-transfer results based on (23), (24) and (29) are tabulated in Table 3 along with the results given by (21) and (28), and the numerical results computed in [4] and [6].

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Résumé—On étudie théoriquement les distributions de températures et la transmission de chaleur dans l'écoulement laminaire au-dessus d'un disque non isotherme en rotation. On suppose que la température de surface du disque varie suivant une loi de puissance en fonction de la distance radiale. On présente des solutions numériques des équations de la couche limite qui traduisent les effets de l'écoulement forcé, de la température de surface non-uniforme et du nombre de Prandtl. Les relations asymptotiques de transmission de chaleur sont données pour de grands et petits nombres de Prandtl.

Zusammenfassung—Die Temperaturverteilungen und der Wärmeübergang für Zwangskonvektionsströmung gegen eine nichtisotherme, rotierende Scheibe wird einer theoretischen Betrachtung unterzogen. Die Oberflächentemperatur der Scheibe soll sich dabei nach einem Potenzgesetz mit dem Radialabstand ändern. Numerische Lösungen der Grenzschichtgleichungen werden angegeben, um den Einfluss der Zwangsströmung auf die nicht einheitliche Oberflächentemperatur und die Prandtl-Zahl anzugeben. Asymptotische Wärmeübergangsbeziehungen für grosse und kleine Prandtl-Zahlen werden mitgeteilt.

Аннотация—Приводится теоретическое решение распределения температур и результаты по теплообмену неизотермического вращающегося диска в вынужденном ламинарном потоке. Предполагают, что температура поверхности диска изменяется по степенному закону вдоль радиуса. Представленные уисленные решения уравнений пограничного слоя указывают на влияние вынужденного потока, неоднородной температуры поверхности и числа Прандтля. Приводятся асимптотические отношения теплообмена для больших и малых чисел Прандтля.